1. (15%) Find the general solution $y(x)$ of the following differential equations:

   (1) (5%) $y' = (y - 9x)^2$

   (2) (10%) $y'' + 4xy' + 4x^2y = 0$

2. (10%) Find the general solution $y(x)$ in terms of Modified Bessel functions of the first kind $I_\nu(x)$ and the second kind $K_\nu(x)$ of order $\nu$ for the differential equation:

   $x^2y'' + \frac{1}{2}xy' - \left( x^2 + \frac{3}{16} \right)y = 0,$

   where $I_\nu(x) = i^{-\nu}J_\nu(ix)$, $i = \sqrt{-1}$, and $K_\nu(x) = \frac{\pi}{2\sin\nu\pi}[I_{-\nu}(x) - I_\nu(x)].$

3. (15%) Solve $y(t)$ of the following integral and differential equations using the Laplace transform method:

   (1) (7%) $y' + e^{-2t} \int_0^t e^{2\tau} y(\tau) d\tau = e^{-t} \int_0^\infty e^{t} \delta(t) dt$, \quad $y(0) = 0$

   (2) (8%) $ty'' + ty' + y = 0$, \quad $y(0) = 0$, \quad $y'(0) = 1$

4. (10%) Show that $\frac{1}{2} \cos t + \cos 2t + \cdots + \cos Mt = \frac{\sin(M + 0.5)t}{2\sin 0.5t}$.

5. (15%) Given the definition of the Laplacian operator in cylindrical coordinates

   $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$, find the steady state temperature distribution in the cylinder with the following boundary conditions,

   1). Temperature $T$ is zero at the radial surface where $r = 2$.

   2). Temperature $T$ is zero at the bottom where $z = 0$, and $T_0$ at the top where $z = 4$.
6. (15%) Now consider the steady-state temperature distribution in a semi-infinite slab

\[ \nabla^2 T = 0, \quad 0 < x < \pi, y > 0 \]

\[ T(0, y) = 0, \quad T(\pi, y) = e^{-y}, y > 0 \]

\[ \frac{\partial T}{\partial y}(x, 0) = 0, \quad 0 < x < \pi \]

Use Fourier cosine transform to solve \( T(x, y) \).

7. (10%) Given \( \vec{F} = xy \hat{i} + y^2 z \hat{j} + z^3 \hat{k} \), calculate \( \iint_{s} \vec{F} \cdot \vec{n} \ ds \), where \( s \) represents the cube confined by planes \( x = \pm 1, \ y = \pm 1, \) and \( z = \pm 1 \).

8. (10%) Consider the symmetric matrix \( A = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix} \)

(a) (5%) Find the matrices \( P \) and \( P^{-1} \) that orthogonally diagonalize the matrix \( A \).

(b) (5%) Find the diagonal matrix \( D \) by actually carrying out the multiplication \( P^{-1}AP \).