Problem 1: (15%)

(a) Compute $A^{10}$ by Cayley-Hamilton theorem. $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (5%)

(b) A nonzero $n \times n$ matrix $A$ is said to be nilpotent of index $m$ if $m$ is the smallest positive integer of which $A^m = 0$. Is the following matrix nilpotent? If nilpotent, what is its index?

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$ (5%)

(c) Find the general solution of the given system by computing $e^{At}$. $X' = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} X$ (5%)

Problem 2: (15%)

Find $\int_C F \cdot dr$. $F(x,y,z) = x^2 y \hat{i} + (x + y^2) \hat{j} + xy^2 \hat{k}$; $C$ is the boundary of the surface shown in figure 1.

![Figure 1](image-url)
Problem 3: (20\%) 
Consider a vibrating elastic bar in figure 2. The boundary at \( x=0 \) and \( x=L \) are called free-end conditions.

Solve the wave equation.
\[
\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < L, \quad t > 0
\]
\[
\left. \frac{\partial u}{\partial x} \right|_{x=0} = 0, \quad \left. \frac{\partial u}{\partial x} \right|_{x=L} = 0
\]
\[
\frac{\partial u}{\partial t} \bigg|_{t=0} = 0
\]

\[u(x,0) = x, \quad \frac{\partial u}{\partial t} \bigg|_{t=0} = 0\]

Problem 4: (20\%) 
Solve the system of differential equations
\[
\frac{d^2 x}{dt^2} + \frac{d^2 y}{dt^2} = t^2
\]
\[
\frac{d^2 x}{dt^2} - \frac{d^2 y}{dt^2} = 4t
\]
\[
x(0) = 0, x'(0) = 0
\]
\[
y(0) = 0, y'(0) = 0
\]

Problem 5: (15\%) 
Find two power series solutions of the differential equation about the ordinary point \( x=0 \).
\[
y'' + 2e^x y = 0
\]

Problem 6: (15\%) 
Solve the differential equation
\[
xy'' + y' - \frac{1}{x} y = \frac{1}{x(x+1)}
\]