Problem 1: (16%) 
Consider the boundary-value problem introduced in the construction of the mathematical model for the shape of rotating string:

\[ T \frac{d^2 y}{dx^2} + \rho \omega^2 y = 0, \quad y(0) = 0, \quad y'(L) = 0. \]

For constant \( T \) and \( \rho \), define the critical speeds of angular rotation \( \omega_n \) as the value of \( \omega \) for which the boundary-value problem has nontrivial solution. Find the critical speeds \( \omega_n \) and the corresponding deflections \( y_n(x) \).

Problem 2: (15%) 
Use the Laplace transform to solve the initial-value problem

\[ y'' + y = \sin t + t \sin t, \quad y(0) = 0, \quad y'(0) = 0. \]

Problem 3: (20%) 
(a) Find the work \( \left( \int \vec{F} \cdot d\vec{r} \right) \) done by the force \( \vec{F}(x, y, z) = (e^x j + xe^y j + yxe^y) k \) acting along the smooth curve \( \vec{r}(t) = t \hat{i} + t^2 \hat{j} + t^3 \hat{k}, \quad 0 \leq t \leq 1 \) (10%)

(b) Find \( \oint_C \vec{F} \cdot d\vec{r} \). \( \vec{F}(x, y, z) = z^2 y \cos xy \hat{i} + z^2 x(1 + \cos xy) \hat{j} + 2z \sin xy \hat{k} \); \( C \) is the boundary of the plane \( z = 1 - y \) shown in Figure 1. (10%)

![Figure 1](image_url)
Problem 4: (15%)
Consider a system

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
-2 & -2 & -2
\end{pmatrix} \begin{pmatrix} x' \\ x \\ x \end{pmatrix}
\]

(a) Find the general solution of the given system using eigenvalues and eigenvectors.

(b) Find the general solution of the given system by diagonalization.

(c) Find the general solution of the given system by computing \(e^u\).

Problem 5: (20%)
Solve the following differential equation defined as:

\[
\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} + x + y + 1 = 0, \text{ where } u = u(x, y)
\]

\[
\left\{ u(0, 0) = 0, \quad \frac{\partial u}{\partial x} \bigg|_{y=0} = 0, \quad \frac{\partial u}{\partial y} \bigg|_{x=0} = 0 \right\}
\]

(Hint: A substitution of \(v(x, y) = \frac{\partial u}{\partial x}\) may be useful for the calculation.)

Problem 6: (14%)
Find all roots of

a. \(\sqrt{1 + i}\)  

b. \(\sqrt{-7 + 24i}\)