1. (30 pts) Suppose \((X, Y)\) have a joint distribution with finite second moments of \(X\) and \(Y\). Denote the means and variances of \(X\) and \(Y\) by \(\mu_1, \mu_2\) and \(\sigma_1^2, \sigma_2^2\), respectively, and let \(\gamma\) be the correlation coefficient between \(X\) and \(Y\).

(1) If \(P(Y = \alpha X + \beta) = 1\), find \((\alpha, \beta)\) in terms of \((\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \gamma)\). (10 pts)

(2) Given two conditional expectations \(E(X|Y) = Y/16 - 3\) and \(E(Y|X) = 4X + 3\), find the value of \((\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \gamma)\) and compute \(E[Var(Y|X)]\). (20 pts)

2. (10 pts) Let \(X\) and \(Y\) be i.i.d. \(N(0, 1)\) random variables, and define \(Z = \min\{X, Y\}\).

(a) Obtain the probability density function of \(Z\). (5 pts)

(b) Prove that \(Z^2 \sim \chi_1^2\). (5 pts)

3. (10 pts) Let \(X_1, \ldots, X_n\) be a random sample from an exponential population with mean \(\theta\), and \(X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}\) are the order statistics.

\[
\begin{align*}
Y_1 &= nX_{(1)}, \\
Y_2 &= (n - 1)(X_{(2)} - X_{(1)}), \\
Y_3 &= (n - 2)(X_{(3)} - X_{(2)}), \\
& \vdots \\
Y_n &= (X_{(n)} - X_{(n-1)}).
\end{align*}
\]

Show that \(Y_1, Y_2, \ldots, Y_n\) are i.i.d. exponential random variables with mean \(\theta\).

4. (15 pts) Let \((X_1, Y_1), \ldots, (X_n, Y_n)\) be a random sample from a bivariate normal distribution with parameters \(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho\). Consider the hypothesis testing

\[H_0 : \mu_X = \mu_Y \quad \text{vs.} \quad H_1 : \mu_X \neq \mu_Y.\]

(a) Derive the distribution of the random variables \(W_i = X_i - Y_i, i = 1, \ldots, n\). (5 pts)

(b) Show that the above hypothesis testing can be tested with the LRT-based statistic

\[T_W = \frac{\bar{W}}{\sqrt{\frac{1}{n} S_W^2}},\]

where \(\bar{W} = \frac{1}{n} \sum_{i=1}^n W_i\) and \(S_W^2 = \frac{1}{(n-1)} \sum_{i=1}^n (W_i - \bar{W})^2\). (5 pts)

(c) Under \(H_0\), find the distribution of the test statistic \(T_W\). (5 pts)
5. (25 pts) An exponential family is a class of probability density functions (p.d.f.) that can be expressed as

\[ f(x; \theta) = \exp\{\phi(\theta) \cdot K(x) + S(x) + \eta(\theta)\} \]

with some conditions imposed on \( \phi(\cdot) \), \( K(\cdot) \), and \( S(\cdot) \).

(a) Let \( X_1, X_2, \ldots, X_n \) be a random sample from a distribution with p.d.f. \( g(x; \theta) = \theta^2 x \exp(-\theta x), 0 < x < \infty, \theta > 0 \). Argue that \( Y = \sum_1^n X_i \) is a complete sufficient statistic for \( \theta \). (Hint: \( g \) is a member of exponential family.) (10 pts)

(b) Compute \( E(1/Y) \) and find the function of \( Y \) which is the unique unbiased minimum variance estimator of \( \theta \). (15 pts)

6. (10 pts) Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from a distribution having p.d.f.

\[ f(x; \theta_1, \theta_2) = \begin{cases} \frac{1}{2\theta_2}, & \theta_1 - \theta_2 < x < \theta_1 + \theta_2, \\ 0 & \text{elsewhere}, \end{cases} \]

where \(-\infty < \theta_1 < \infty, 0 < \theta_2 < \infty\). Let \( X_{(1)} < X_{(2)} < \ldots < X_{(n)} \) be the order statistics. What is the joint p.d.f. of \( X_{(1)} \) and \( X_{(n)} \)?